Rotational Motion and Energy

Summary and Exercises

1. **Rotation**

Period of rotation

In circular motion and rotation we define the time it takes for one complete revolution as the Period T (unit s). E.g. if one revolution takes 5 s then we say that $T = 5$ s.

Frequency of rotation

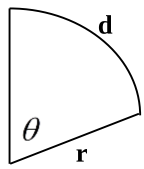
The frequency f of this motion is defined as the number of revolutions per second (unit s^{-1} or Hz). So frequency is the inverse of Period, hence $f = \frac{1}{T}$.

In the example above $f = \frac{1}{T} = \frac{1}{5} = 0.2$ Hz.

In mechanical engineering we often use Revolutions per minute (RPM) but that is not an SI unit. Because there are 60 seconds in a minute we can convert RPM to Hz by dividing by 60.

Example: 1200 RPM is equal to $\frac{1200}{60} = 20$ Hz.

2. **Radian as Angular measure**



The radian is a measure for angular displacement. It is defined in such a way that if we multiply the angle in radian with the radial distance, we obtain the displacement d along the arc. Hence $d = r\theta$.

The circumference of a circle is $2\pi r$ therefore the angle of a full circle is 2π radians

$$d = r\theta \Rightarrow 2\pi r = r \times 2\pi.$$

The radian is **not a unit**. Re-arrange $d = r\theta$ as $\theta = \frac{d}{r}$ and you see that the radian actually is a ratio of two distances, hence it has no unit. It simply is a **measure** for angular displacement.

Exercises

a. Convert these angles to radians. Leave π in your answer:
   90°, 60°, 120°, 270°, 2 revs, 1.5 revs

b. Convert these angular speeds into rad s^{-1}:
   2 revs per second, 4 revs in 0.2 s, 10 Hz, 600 RPM, time period $T = 5$ ms

c. A sling consists of a stone spinning at the end of a string at 2.6 times per second around a circle with a radius of 1.5 m.
   i. Calculate the stone's angular velocity ω
   ii. Calculate the stone's linear velocity

d. Calculate the linear velocity of the edge of a 25 cm diameter grinding wheel spinning at 3600 RPM.

e. A 25 cm radius wheel is set rolling along the ground at 0.65 ms^{-1}.
   i. Calculate its angular velocity
   ii. It rolls 2.8 m along the ground before stopping. Calculate how many radians it rotates through
   iii. How long does its journey take?

3. Rotational Kinematics

Equations of motion

In linear kinematics we used the concepts of displacement (d), velocity (v) and acceleration (a). For studying rotations we introduce similar concepts of angular displacement (θ), angular velocity (ω) and angular acceleration (α).

The SI units are: θ (rad); ω (rad s^{-1}); α (rad s^{-2}).

We already saw that $d = r\theta$ to relate angular displacement to linear displacement. Similarly we have $v = r\omega$ and $a = r\alpha$ so there is a strong symmetry between linear and rotational motion.

Now we can re-write the linear kinematic equations of motion in angular form:

$$v_f = v_i + at \quad \Rightarrow \quad \omega_f = \omega_i + \alpha t$$

$$v_f^2 = v_i^2 + 2ad \quad \Rightarrow \quad \omega_f^2 = \omega_i^2 + 2\alpha\theta$$

$$d = v_i t + \frac{1}{2}at^2 \quad \Rightarrow \quad \theta = \omega_i t + \frac{1}{2}\alpha t^2$$

When solving problems in kinematics start with making up a budget of which quantities are known and which you want to calculate. It is then easy to decide which of these formula's to use.

Angular Frequency

Above we defined frequency (f) as revolutions per second. Because there are 2π radians in one revolution, we can express **Angular frequency** ω as $\omega = 2\pi f = \frac{2\pi}{T}$ where T is the Period (s).

This Angular Frequency is the same as the angular velocity ω (unit is rad s^{-1}).

Compare Linear velocity (for circular motion) $v = \frac{2\pi r}{T}$ with Angular velocity $\omega = \frac{2\pi}{T}$ (remember $v = r\omega$).

Exercises

a. A bike wheel accelerates from an angular speed of 10 rad s-1 to 210 rad s-1 over a time of 5.0 s.
   i. Calculate the wheel's angular acceleration
   ii. Calculate how many radians it rotates
   iii. Convert the radians to the number of revolutions
   iv. How long does it take to reach a speed of 90 rad s-1?
   v. What is the angular velocity when it has rotated through 200 rad?

b. A turntable slows down from 8.0 rad s^{-1} to 4.0 rad s^{-1}. During this time it rotates exactly seven times.
    i. Calculate the angular displacement during this time
    ii. Calculate the angular acceleration
    iii. Calculate the time is takes
    iv. Calculate how many turns it has made after 5 s.

c. A car with 32 cm radius wheels is travelling at 16 ms^{-1}.
    i. Calculate the angular speed of the wheels
    ii. The car slows down to a stop. AS this happens, the wheels rotate 25 times. Calculate the wheel's angular displacement
    iii. Calculate the wheels angular acceleration
    iv. Calculate the car's linear acceleration
    v. Calculate the time it takes to stop
    vi. How far does the car travel while braking?

4. Rotational Dynamics

Torque is "Angular Force" caused by a force couple or by a force *F* at a distance *r* from the leverage point: $\tau = Fr$ (unit Nm).
Similar to Newton's second law: $F = ma$ where a force gives a linear acceleration to a mass, a torque can create angular acceleration:

$$\tau = I\alpha .$$

The proportionality constant I is **Rotational Inertia** (Unit kgm^2).

Exercises

a. A tangential force of 2.5 N acts on a turntable with a radius of 32 cm. The turntable takes 1.2 s to complete the first revolution from rest.
    i. What is the angular displacement for one revolution?
    ii. Use an equation of motion to find the angular acceleration
    iii. Calculate the torque acting on the turntable
    iv. What is the rotational inertia of the turntable?

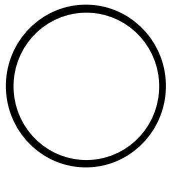
b. A cord is wrapped around the axle of a wheel. A 6.5 kg mass on the cord accelerates the wheel from rest. The angular acceleration is 1.8 rad s^{-2}. The radius of the axle is 0.12 m. (Use g=10 ms^{-2})
    i. Calculate the tension force in the cord
    ii. Calculate the torque acting on the wheel
    iii. Calculate the rotational inertia of the wheel
    iv. The torque is applied for 5 s. Calculate the angle turned through in this time

c. A student uses a rowing machine and must pull on a chain wrapped around the axle of a flywheel, causing the flywheel to rotate. During the first pull the flywheel is uniformly accelerated from rest. This acceleration occurs over a distance (along the chain) of 0.48 m and takes a time of 0.85 s. The radius of the axle is 1.2 cm and the radius of the flywheel is 25 cm.

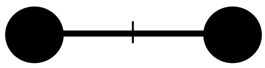
   i. Show that the angle the flywheel turns through is 40 rad.
   ii. Calculate the angular acceleration
   iii. The average force that is applied to the chain is 185 N. Calculate the torque applied to the axle
   iv. Calculate the flywheel's rotational inertia.
   v. If an axle is used with a larger radius, will it be harder or easier to achieve the same angular acceleration? Explain.

5. Rotational Inertia and Mass distribution

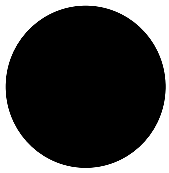
Mass is a single constant for any object;
Rotational Inertia depends on the **mass distribution** over the object: $I = \sum mr^2$.
Therefore objects with the same mass but different shape will have different rotational inertia.
Rotational inertia is defined with respect to a centre of rotation. Each mass particle of an object contributes to the rotational inertia with mr^2, where m is the mass of the particle and r is the distance from the centre of rotation. The total rotational inertia then is the sum of all contributions: $I = \sum mr^2$.



- For a hoop (neglect its thickness) all particles are at the same distance to the centre. Therefore the rotational inertia of a hoop is $I = \sum mr^2 = Mr^2$ where M is the total mass.



- For a dumbbell (neglect the connecting rod) the rotational inertia with respect to its centre is $m_1r^2 + m_2r^2 = (m_1 + m_2)r^2$



- A solid disk of same mass and radius as a hoop, will have a smaller rotational inertia than the hoop, because many mass particles are closer to the centre than the radius. The rotational inertia of a solid disk is approximately $0.4Mr^2$, where M is the total mass and r its radius.

Exercises

a. Using the equation $\tau = I\alpha$ show that the unit for Rotational Inertia is kgm^2.

b. A dumbbell consists of two 1.5 kg masses that are 1.2 m apart. Calculate its rotational inertia with respect to its centre.

c. An ice skater stands straight up and has a certain rotational inertia about her vertical axis. She now spreads her arms wide. What happens to her rotational inertia? Explain.

6. **Angular Momentum**
   compare with $p = mv$: $L = I\omega$ (unit kgm^2s^{-1})

7. **Conservation of Angular Momentum**
   Angular Momentum is conserved when there is no external torque.

   Examples: rotating ice skater, somersault, air masses in atmosphere of rotating earth.
   If I changes because mass distribution changes, angular velocity must also change to conserve Angular Momentum.

8. **Angular Momentum of a particle**
   Angular Momentum of a point mass in circular motion about a centre is simply the Linear Momentum times the radius: $L = mvr$

9. **Rotational Kinetic Energy**
   Work is Torque times Angular displacement: $W = \tau\theta$ (compare $W = F \times d$)

   Rotational Kinetic Energy: $E_{rot} = \frac{1}{2}I\omega^2$ (compare $E_{kin} = \frac{1}{2}mv^2$)

10. **Rolling**

    Energy in rolling is sum of linear and rotational energy: $E_{rolling} = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

Given two cylinders which look alike but have different Rotational Inertia:

Cylinder 1	Cylinder 2
m = 0.750 kg	m = 0.750 kg
r = 0.120 m	r = 0.120 m
I = 0.00750 kgm^2	I = 0.0150 kgm^2

Both cylinders are rolled down a slope with a height difference of 1.20 m.
At which speed v do they arrive at the bottom of the slope?

First we calculate the loss of gravitational potential energy:
$E_{gravpot} = m \times g \times h = 0.750 \times 9.80 \times 0.120 = 8.82$ J

This energy is converted into Linear and Rotational kinetic energy. So at the bottom of the slope
$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = 8.82$.

We can solve speed v if we first substitute $v = r\omega$ or $\omega = \frac{v}{r}$ into this expression:

$$\frac{1}{2}mv^2 + \frac{1}{2}I\frac{v^2}{r^2} = \frac{1}{2}v^2\left(m + \frac{I}{r^2}\right) = 8.82 \text{ or } \boxed{v = \sqrt{\frac{8.82}{\frac{1}{2}\left(m + \frac{I}{r^2}\right)}}}$$

Thus we find for cylinder 1: $v = \sqrt{\dfrac{8.82}{\dfrac{1}{2}\left(0.750 + \dfrac{0.00750}{0.120^2}\right)}} = 3.73$ ms^{-1}

and for cylinder 2: $v = \sqrt{\dfrac{8.82}{\dfrac{1}{2}\left(0.750 + \dfrac{0.0150}{0.120^2}\right)}} = 3.13$ ms^{-1}

So the cylinder with the *largest* rotational inertia is the *slowest*.

11. Comparison Linear and Rotational Motion

Physical Quantity	Linear		Relationship	Rotational	
	Unit	Symbol		Symbol	Unit
Displacement	m	d	$d = r\theta$	θ	rad
Velocity	ms^{-1}	v	$v = r\omega$	ω	rads^{-1}
Acceleration	ms^{-2}	a	$a = r\alpha$	α	rads^{-2}
Equations of Motion	$v_f = v_i + at$ $v_f^2 = v_i^2 + 2ad$ $d = v_i t + \frac{1}{2}at^2$			$\omega_f = \omega_i + \alpha t$ $\omega_f^2 = \omega_i^2 + 2\alpha\theta$ $\theta = \omega_i t + \frac{1}{2}\alpha t^2$	
Force / Torque	N	F	$\tau = Fr$	τ	Nm
Newton's law	$F = ma$			$\tau = I\alpha$	
Mass / Rotational Inertia	kg	m	$I = \sum mr^2$	I	kgm^2
Work	J (Nm)	$W = Fd$		$W = \tau\theta$	J (Nm)
Kinetic Energy	J	$E_{kin}^{lin} = \frac{1}{2}mv^2$		$E_{kin}^{rot} = \frac{1}{2}I\omega^2$	J
Momentum	kgms^{-1}	$p = mv$	$L = mvr$ (particle)	$L = I\omega$	kgm^{2}s^{-1}

Remember the (confusing) units for:

Linear Momentum	kgms^{-1}
Angular Momentum	kgm^{2}s^{-1}
Rotational Inertia	kgm^2