Calculus Year 13 (Level 8)

Summary 7

Complex Numbers

Complex Numbers are an extension of Real numbers to be able to always find roots of any polynomial (see handout "Basic Set Theory and Number Systems of Algebra"). It is convenient to imagine that complex numbers represent points (co-ordinates) in the **Argand plane**, where the horizontal axis is the set **R** and the imaginary part is measured along the vertical axis.

$i^2 = -1$ or $i = \sqrt{-1}$

Rectangular form: $z = x + iy$; $x = \text{Re}(z)$ and $y = \text{Im}(z)$ $\{x, y \in R\}$

Polar form $z = rcis(\theta) = r(\cos\theta + i\sin\theta)$ r is **Modulus**, θ is **Argument**

Conversions: $R \to P$: $r = \sqrt{x^2 + y^2}$; $\cos\theta = \frac{x}{r}$; $\sin\theta = \frac{y}{r}$; and

$P \to R$: $x = r\cos\theta$; $y = r\sin\theta$

Addition and Subtraction: $(x_1 + iy_1) \pm (x_2 + iy_2) = (x_1 \pm x_2) + i(y_1 \pm y_2)$

Multiplication:

$$(x_1 + iy_1) \bullet (x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

$$r_1 cis\theta_1 \bullet r_2 cis\theta_2 = r_1 r_2 cis(\theta_1 + \theta_2)$$

Conjugate: $\bar{z} = x - iy$ (where $z = x + iy$) (mirror about R-axis in Argand plane)

$\bar{z} = rcis(-\theta)$ (where $z = rcis\theta$)

Division:

$$\frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{1}{x_2^2 + y_2^2}(x_1 + iy_1)(x_2 - iy_2)$$

$$\frac{r_1 cis\theta_1}{r_2 cis\theta_2} = \frac{r_1}{r_2} cis(\theta_1 - \theta_2)$$

Modulus: $|z| = r = \sqrt{x^2 + y^2}$

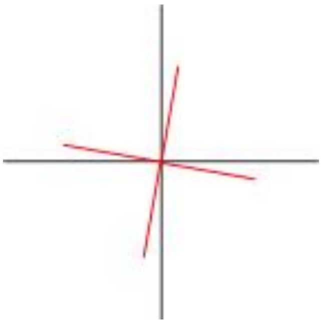
Raise to **power**: $z^n = (rcis\theta)^n = r^n cis(n\theta)$; $n \in I$ (**De Moivre**)

Solve quadratic **Polynomials**: use $i = \sqrt{-1}$ for negative square roots,

Example: $x^2 - 4x + 5 = 0$ has solutions

$$x_{1,2} = \frac{4 \pm \sqrt{(-4)^2 - 4(5)}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2\sqrt{-1}}{2} = 2 \pm i$$

Conjugate Root Theorem: Complex roots of any polynomial over **R** come in conjugate pairs, hence if $P(x)$ is a polynomial over **R** and z is a root, then $\bar{z}$ is also a root.



Complex Roots of Polynomials over **C**: $z^n = rcis\theta$ has *n* solutions $r^{\frac{1}{n}}cis\left(\frac{\theta}{n}+\frac{2\pi}{n}\right)$ within the interval $-180 < \alpha < 180$

Example: $z^4 = (5cis(-36.9))^4$ has roots $5^{\frac{1}{4}}cis\left(\frac{-36.9}{4}+\frac{360}{4}\right)$

giving the solutions

$1.495cis(-99.2), 1.495cis(-9.2), 1.495cis(80.8), 1.495cis(170.8)$

Note that these solutions are symmetric about the origin in the Argand plane.