Calculus Year 13 (Level 8)

Stationary Points

A function **decreases** if for increasing x the function value decreases and a function **increases** if for increasing x the function value increases. This can be seen from the first derivative:

$y' < 0 \Rightarrow$ y is decreasing and $y' > 0 \Rightarrow$ y is increasing

See the graph:

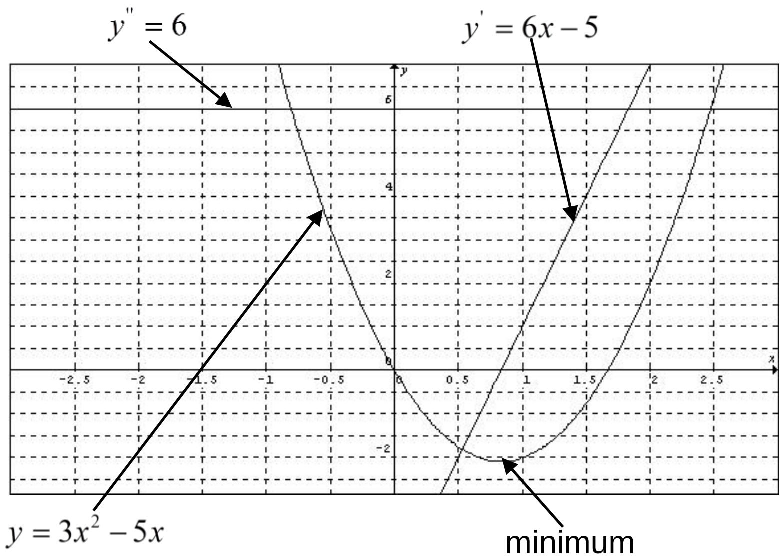
for $x < \frac{5}{6}$ the tangent to the parabola has a negative gradient, hence y' is negative.

For $x > \frac{5}{6}$ the tangent has a positive gradient, hence y' is positive.

At the point $x = \frac{5}{6}$ the gradient is zero, the tangent is horizontal, hence $y' = 0$.

In summary:

$y' < 0 \Rightarrow$	y decreases
$y' > 0 \Rightarrow$	y increases
$y' = 0 \Rightarrow$	y is stationary (turning point)



At a **turning point** how do we know if there is a maximum or a minimum?

That depends on how the tangent to the curve behaves on either side of the turning point. In the graph the tangent to the parabola is increasing for increasing values of x; you can say that the it rotates counter clockwise if x increases around the turning point. This is illustrated by the fact that the derivative y' is increasing. Thus the second derivative must be positive. Hence for a **positive second derivative** the original function reaches a **minimum** at the turning point. Similarly when the second derivative is negative the turning point is a maximum.

In summary:

y'	y''	Hence y:
0	<0	maximum
0	>0	minimum

There is yet another type of stationary point, i.e. when the second derivative is zero. This is illustrated in the graph below.

At x = 2 the second derivative is zero. This means that the first derivative has a turning point just like we discussed before. But this means that the tangent to the original curve at this point changes direction of rotation, in other words (the direction of curvature of the original function changes from concave to convex (or the other way around). This is a **point of inflection**.

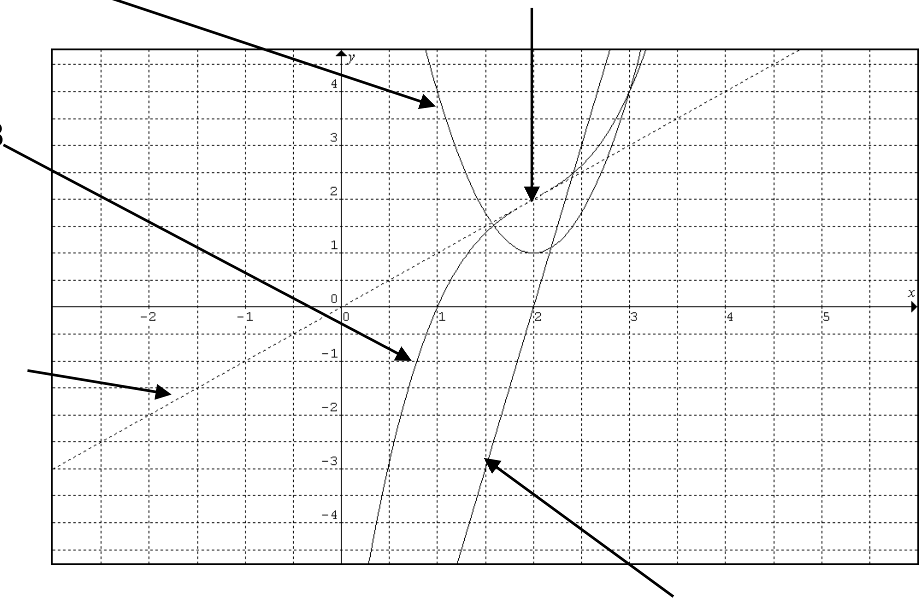
In general the first derivative does not need to be zero at a point of inflection. Only the condition $y''=0$ needs to be satisfied to have a point of inflection.

First derivative:
$y'=3x^2-12x+13$

point of inflection

Original function:
$y=x^3-6x^2+13x-8$

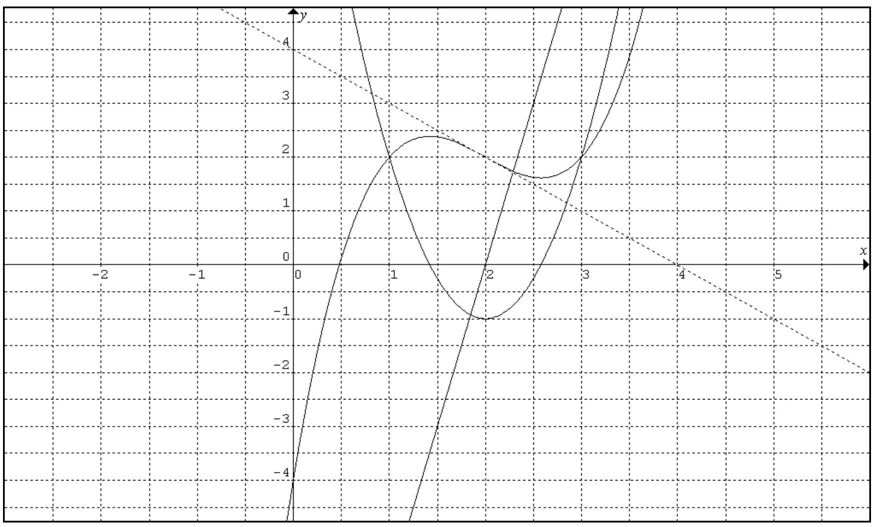
Tangent line
at point of inflection



Second derivative: $y''=6x-12$

Another example is given here:

$y=x^3-6x^2+11x-4$
$y'=3x^2-12x+11$
$y''=6x-12$



Verify yourself that there is a maximum at the point $x=1.42$ and a minimum at the point $x=2.58$.
Verify also that the condition $y''=0$ is satisfied at the point $x=2$ hence there is a point of inflection.
Graph these functions yourself in Graphmatica to verify these situations.
In summary:

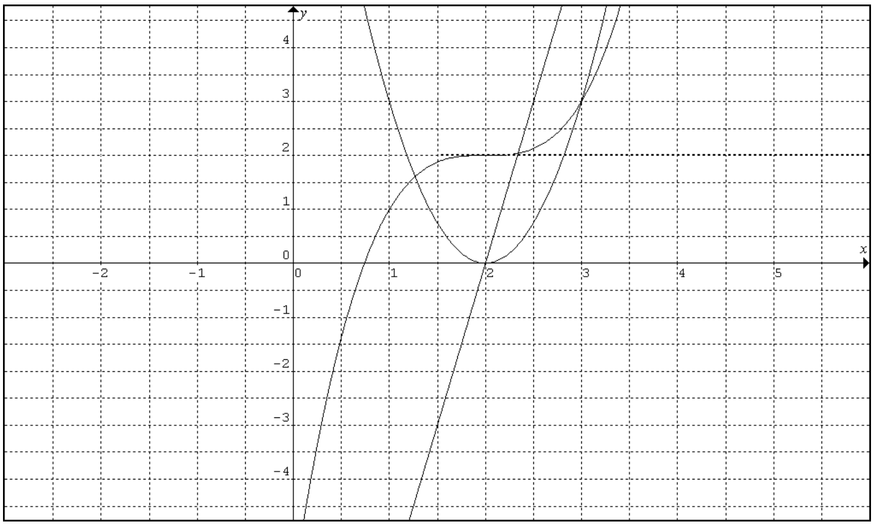
$$y' = 0 \rightarrow turning\ point \quad \begin{cases} y'' > 0 \rightarrow minimum \\ y'' < 0 \rightarrow maximum \end{cases}$$

$$y'' = 0 \rightarrow point of\ inflection$$

Remark 1:
We saw that the condition $y''=0$ is sufficient for a point of inflection. But in some situations the first derivative **could** also be zero.
Initially you would think that because $y'=0$ there is a turning point (maximum or minimum) but once you check out the second derivative and find that this is also zero the conclusion is that there is a point of inflection here and no turning point.

An example of this situation is depicted below. You see that the tangent line at the point of inflection is horizontal, hence the first derivative is zero at this point. But the second derivative is also zero so this is a point of inflection and no turning point.

$$y = x^3 - 6x^2 + 12x - 4$$
$$y' = 3x^2 - 12x + 12$$
$$y'' = 6x - 12$$



Remark 2:
In exceptional cases (especially higher (even) powers of x you could find that both $y'=0$ and $y''=0$ and that there yet is a maximum or minimum rather than a point of inflection. The only way to find that out if you don't draw the graph is to check values on either side of the point.

E.g. $y=x^4;\ y'=4x^3;\ y''=12x^2$ are all zero for $x=0$ yet that is a minimum.

E.g. $y=-x^4;\ y'=-4x^3;\ y''=-12x^2$ are all zero for $x=0$ yet this is a maximum.

Verify this with graphmatica.